# Geometric properties of the FF-curve, Part II

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List of symbols

- $p$ a fixed prime
- $E/\mathbb{Q}_p$ finite extension (E.g. $E = \mathbb{Q}_p$)
- $\mathcal{O}_E \subset E$ ring of integers (E.g. $\mathcal{O}_E = \mathbb{Z}_p$)
- $\pi \in \mathcal{O}_E$ uniformizer (E.g. $\pi = p$)
- $\mathbf{F}_q = \mathcal{O}_E/(\pi)$ the residue field of $\mathcal{O}_E$ (E.g. $\mathbf{F}_p$)
- $F$ = an algebraically closed cvf extension of $\mathbf{F}_q$ with non-trivial non-archimedean rank-1-valuation $v$ (E.g. $F = \widehat{\overline{\mathbf{F}_p((T))}}$, value group $v(F^*) = \mathbb{Q}$)
- $\mathcal{O}_F$ the ring of integers of $F$ [3] (E.g. $\mathcal{O}_F = \widehat{\overline{\mathbf{F}_p[[T]]}}^F$)
- $\mathfrak{m}_F$ the maximal ideal of $\mathcal{O}_F$
- $\varpi \in \mathfrak{m}_F - \{0\}$ pseudo-uniformizer (E.g. $\varpi = T \in \mathfrak{m}_F$)
- $k = \mathcal{O}_F/\mathfrak{m}_F$ (E.g. $k = \overline{\mathbf{F}}_p$)

Further list of symbols, towards the schematic Fargues-Fontaine curve:

- $\mathbf{A}_{\inf} = \mathbf{A}_{\inf,E,F} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \{\sum_{i\geq 0}[x_i]\pi^i \mid x_i \in \mathcal{O}_F\} \subset W_{\mathcal{O}_E}(F)$
- $B^b = \mathbf{A}_{\inf}[\frac{1}{\pi}, \frac{1}{[\varpi]}] = \{\sum_{i\gg -\infty}^{\infty}[x_i]\pi^i \mid x_i \in F, \sup_i |x_i| < \infty\} \subset W_{\mathcal{O}_E}(F)[\frac{1}{\pi}]$ [5]
- $v_r$ the valuation associated to $r \in \mathbb{R}$ [6]
- $|\cdot|_\rho$ the Gauss norm [7]
- $B_I$ the completion of $B^b$ with respect to valuations $v_r$ for all $r \in I$ [8]
- $B = B_{(0,\infty)}$ [9]
- $P = P_{E,F,\pi} = \bigoplus_{d\in\mathbb{N}} B^{\varphi=\pi^d}$
- $X = X_{E,F,\pi} = \mathrm{Proj}(P_{E,F,\pi})$, the schematic Fargues-Fontaine curve

Further list of symbols, towards the adic Fargues-Fontaine curve:

- $\mathrm{Prim}_d = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\inf} \mid x_i \in \mathcal{O}_F;\ x_0 \in \mathfrak{m}_F \setminus \{0\},\ x_1, \dots, x_{d-1} \in \mathfrak{m}_F,\ x_d \in \mathcal{O}_F^*\}$. In particular,
  $\mathrm{Prim}_0 = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\inf} \mid x_0 \in \mathcal{O}_F^*\} = \mathbf{A}_{\inf}^*$
  $\mathrm{Prim}_1 = \{x = \sum_{i=0}^{\infty}[x_i]\pi^i \in \mathbf{A}_{\inf} \mid x_0 \in \mathfrak{m}_F \setminus \{0\}, x_1 \in \mathcal{O}_F^*\}$
  $= \{[x_0] + u\pi \in \mathbf{A}_{\inf} \mid x_0 \in \mathfrak{m}_F \setminus \{0\},\ u \in \mathbf{A}_{\inf}^*\}$
  $= \{\text{distinguished elements}\} \setminus \{u\pi \in \mathbf{A}_{\inf} \mid u \in \mathbf{A}_{\inf}^*\}$
- $|Y|_{[0,\infty)} = \{(d) \subset \mathbf{A}_{\inf} \mid d \text{ is a distinguished element}\}$
  $= \{([x_0] + u\pi) \subset \mathbf{A}_{\inf} \mid x_0 \in \mathfrak{m}_F, u \in \mathbf{A}_{\inf}^*\}$
- $|Y| = |Y|_{[0,\infty)} \setminus \{(\pi)\}$
  $= \mathrm{Prim}_1 / \mathbf{A}_{\inf}^*$
  $\overset{!}{=} \{(C, \iota) \mid C/E \text{ non-arch, complete, alg closed}, \iota : \mathcal{O}_C^\flat \simeq \mathcal{O}_F\}$

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## Geometric properties, Part II.

1. FES of p-adic Hodge
2. regular curve
3. Cohomology of line bundles
4. The Picard.
5. The étale fundamental group.

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## § 3. Cohomology of line bundles.

Recall what we mean by $\mathcal{O}(1)$ on a non-projective scheme $X$?

The coherent sheaf $\tilde{M}$ on $X$ corresp to the gded module

$$M = \bigoplus_{n\in\mathbb{Z}} M_n = \bigoplus_{n\in\mathbb{Z}} B^{\varphi = p^{n+1}}$$

Construction of "projective tilde" $\longrightarrow$ for $t = \log([\varepsilon]) \in B^{\varphi=p} = P_1$

$$\tilde{M}(D^+(t)) = M[\tfrac{1}{t}]_0 = P[\tfrac{1}{t}]_1 = B[\tfrac{1}{t}]^{\varphi=p}.$$

In particular,

In particular,

$$\mathcal{O}(1)|_{D^+(t)} = B[\tfrac{1}{t}]^{\varphi=id}\cdot t = \mathcal{O}\cdot t.$$

$\rightarrow$ is a line bundle.

Denote $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$.

Thm (cohomology)

(1) $H^0(X, \mathcal{O}(n)) = B^{\varphi=p^n}$. In part.

- $= \mathbb{Q}_p$ when $n=0$.
- $= m_F$ when $n=1$
- $= 0$ when $n<0$

"degenerate" (Lubin-Tate) In the sense that used the formal gp law $\mathbb{G}_m$

$\mathbb{G}_m := (1+X)(1+Y)-1 \in \mathcal{O}_E[[X,Y]]$

(corresp to. $f=(1+x)^p-1 \in \mathcal{F}_\pi$)

(2) $H^i(X, \mathcal{O}(n)) = 0$ when $i>0$ & $n\geq 0$.

Rmk ($\mathbb{P}^1_{\mathbb{Q}}$-analog)

Lemma $\mathcal{O}(1) \cong \mathcal{O}(x)$ where $x \in V^+(t)$

Pf. By construction of $\mathcal{O}(1)$, $\exists$ a map.

$$\mu_1 : B^{\varphi=p} \longrightarrow \Gamma(X, \mathcal{O}(1))$$

locally on $D^+(f)$ with $f \in B^{\varphi=p}$, given by

$$B^{\varphi=p} \ni a \longmapsto \frac{a}{f} \in \Gamma(D^+(f), \mathcal{O})$$

Transition function on $D^+(f) \cap D^+(g)$ with $g \in B^{\varphi=p}$

is $\frac{g}{f}$ $\rightsquigarrow$ glue to $\mathcal{O}(1)$

Now $\mu_1(t) \in \Gamma(X, \mathcal{O}(1)) \longrightarrow$ gives an isomorphism

Now $\mu_1(t) \in \Gamma(X, \mathcal{O}(1)) \longrightarrow$ gives an isomorphism of $\mathcal{O}(1)$ with $\mathcal{O}(x)$. because they are both dual to the ideal sheaf of $x \in X$:

$$\mathcal{O} \xrightarrow{t} \mathcal{O}(1)$$

$$\begin{array}{ccccccccc} \leadsto 0 \to & \mathcal{O}(1)^\vee & \xrightarrow{t^\vee} & \mathcal{O} & \longrightarrow & k(x) & \to & 0 \\ & \| & & \| & & \| & & \\ 0 \to & I(x) & \hookrightarrow & \mathcal{O} & \longrightarrow & k(x) & \to & 0 \end{array}$$

$$\Rightarrow \quad \mathcal{O}(1) = I(x)^\vee = \mathcal{O}(x). \qquad \square$$

Note that we did not use $\mathcal{O}(-1) = \mathcal{O}(1)^\vee$
(but it holds, based on Goal 1 and the pairing $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$
For this don't need of fin type.)
but used the rather harmless $(\mathcal{L}^\vee)^\vee \cong \mathcal{L}$ that simply holds true in any Picard.

<u>Pf of the thm</u>

we prove only for $n \geq 0$. $\quad t = \log([\varepsilon]) \in B^{\varphi = p}$

$\exists$ map

$$\mu_n : B^{\varphi = p^n} \longrightarrow \Gamma(X, \mathcal{O}(n))$$

locally on $D^+(f)$ with $f \in B^{\varphi = p}$:

$$a \longmapsto \frac{a}{f^n}$$

$\xrightarrow{\frac{g^n}{f^n}}$ glue to $\mathcal{O}(n)$.

Consider

$$\begin{array}{ccccccc} 0 \to B^{\varphi = p^n} & \xrightarrow{t} & B^{\varphi = p^{n+1}} & \longrightarrow & B^{\varphi = p^{n+1}} / t B^{\varphi = p} & \to 0 \\ \downarrow \mu_n & & \downarrow \mu_{n+1} & & \cong \downarrow \nu & \\ 0 \to H^0(X, \mathcal{O}(n)) & \xrightarrow{t} & H^0(X, \mathcal{O}(n+1)) & \xrightarrow{\alpha} & H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n)) & \to \cdots \end{array}$$

<u>Claim</u> For $n \geq 0$

$$\nu : B^{\varphi = p^{n+1}} / t B^{\varphi = p} \longrightarrow H^0(X, \mathcal{O}(n+1)/t\,\mathcal{O}(n))$$

is an iso.

<u>Consequences</u> $\Rightarrow \alpha$ is surj.

(1) $0 \to H^0(X, \mathcal{O}(n)) \xrightarrow{t} H^0(X, \mathcal{O}(n+1)) \xrightarrow{\alpha} H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n)) \to 0$. exact

(1) $0 \to H^0(X, \mathcal{O}(n)) \xrightarrow{t} H^0(X, \mathcal{O}(n+1)) \xrightarrow{\alpha} H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n)) \to 0$. exact.

(2). $H^1(X, \mathcal{O}(n)) \xrightarrow[\cong]{t} H^1(X, \mathcal{O}(n+1))$

(bc. $H^1(X, \mathcal{O}(n+1)/t\mathcal{O}(n)) = 0$

$\mathcal{O}(n+1)/t\mathcal{O}(n)$ is a skyscraper supported at $x$. )

(3) $\operatorname{Ker}(\mu_n) \xrightarrow[\cong]{t} \operatorname{Ker}(\mu_{n+1})$ (snake lemma).

(4) $\operatorname{Cok}(\mu_n) \xrightarrow[\cong]{t} \operatorname{Cok}(\mu_{n+1})$ (snake lemma)

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<u>Consequences</u> $\Rightarrow$ Thm

Because $V^+(\underbrace{t}_{\mu_1(t) \in \Gamma(X, \mathcal{O}(1))}) = \{x\}$ $\quad$ [ $t \to \exists!\, y \in |Y|$ $\to x = x_y$ ]

$\rightsquigarrow \quad \mathcal{O}(nx)|_{D^+(t)} = \mathcal{O}_{D^+(t)} \cdot \frac{1}{t^n}$ $\quad$ [transition maps are natural inclusions $\mathcal{O} \cdot \frac{1}{t^n} \hookrightarrow \mathcal{O} \frac{1}{t^{n+1}}$]

$\rightsquigarrow \quad \operatorname*{colim}_{\substack{n \geq 0 \\ \text{via } t}} \mathcal{O}(n)|_{D^+(t)} = \operatorname*{colim}_{n \geq 0} \mathcal{O}_{D^+(t)} \cdot \frac{1}{t^n}$

$= j_* j^* \mathcal{O}_{D^+(t)}$

where $j : D^+(t) \hookrightarrow X$ open imm.

$\Rightarrow \quad \operatorname*{colim}_{\substack{n \geq 0 \\ \text{via } t}} H^0(X, \mathcal{O}(n)) = H^0(X, \operatorname*{colim}_{\substack{n \geq 0 \\ \text{via } t}} \mathcal{O}(n))$

$= H^0(D^+(t), \mathcal{O}_{D^+(t)})$

Now take colimit of $\mu_n : B^{\varphi = p^n} \to \Gamma(X, \mathcal{O}(n))$

Now take colimit of $\mu_n : B \quad \to H^0(X, \mathcal{O}(n))$

$$\rightsquigarrow \quad B[\tfrac{1}{t}]^{\varphi=id} \underset{\text{via } t}{\overset{\uparrow}{=}} \operatorname*{colim}_{n\geq 0} B^{\varphi=p^n} \xrightarrow[\sim]{\mu} \operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t.}} H^0(X, \mathcal{O}(n)) = H^0(D^+(t), \mathcal{O}_{D^+(t)})$$

Canonical iso

$$B^{\varphi=p^n} \xrightarrow{t} B^{\varphi=p^{n+1}}$$

$$B^{\varphi=p^{n+1}} \supset \operatorname{Im}\left(B^{\varphi=p^n} \xrightarrow{t} B^{\varphi=p^{n+1}}\right) = tB^{\varphi=p^n} \supset \cdots \supset t^{n+1} B^{\varphi=id}$$

$$\rightsquigarrow B^{\varphi=p^{n+1}} \cdot \frac{1}{t^{n+1}} \supset B^{\varphi=id}$$

$$B[\tfrac{1}{t}]^{\varphi=id} = B^{\varphi=id} \cup B^{\varphi=p} \cdot \frac{1}{t} \cup B^{\varphi=p^2} \cdot \frac{1}{t^2} \cup B^{\varphi=p^3} \cdot \frac{1}{t^3} \cup \cdots = \operatorname*{colim}_{\substack{n\geq 0\\ \text{natural inclusion}}} B^{\varphi=p^n} \cdot \frac{1}{t^n} = \operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t}} B^{\varphi=p^n}$$

On the other hand

$$\ker \mu = \operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t.}} (\ker \mu_n)$$

$$\operatorname{Cok} \mu = \operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t}} (\operatorname{Cok} \mu_n)$$

constant sequences by (3) & (4)

$$\Rightarrow \forall n \geq 0 \qquad 0 = \operatorname{colim} \ker(\mu_n) = \ker \mu_n$$

$$0 = \operatorname{colim} \operatorname{Cok}(\mu_n) = \operatorname{Cok} \mu_n.$$

$\Rightarrow \forall n \geq 0$ $\mu_n$ is an iso $\Rightarrow$ (1).

For (2)

Consequence (2)

$$\operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t}} H^1(X, \mathcal{O}(n)) = H^1(X, \mathcal{O}(n))$$

Same reasoning as in (1) affine vanishing

$$\operatorname*{colim}_{\substack{n\geq 0\\ \text{via } t}} H^1(X, \mathcal{O}(n)) = H^1(D^+(t), \mathcal{O}_{D^+(t)}) = 0$$

$$\Rightarrow H^1(X, \mathcal{O}(n)) = 0 \qquad \boxtimes$$

Pf of the Claim

$$v: B^{\varphi=p^{n+1}}/tB^{\varphi=p^n} \xrightarrow{\cong} H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n))$$

Pf. First observe that for $f \in B^{\varphi=p}$ s.t. $x \in D^+(f)$.

Have a comm diagram:

$$\begin{array}{ccc}
B^{\varphi=p}/tB^{\varphi=1} & \xrightarrow{f^n} & B^{\varphi=p^{n+1}}/tB^{\varphi=p^n} \\
\downarrow v & & \downarrow v \\
H^0(X, \mathcal{O}(1)/t\mathcal{O}) & \xrightarrow[\cong]{f^n} & H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n))
\end{array}$$

why is iso ?

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O} & \xrightarrow{t} & \mathcal{O}(1) & \to & \mathcal{O}(1)/t\mathcal{O} & \to & 0 \\
 & & \downarrow f^n & & \downarrow f^n & & \downarrow f^n & & \\
0 & \to & \mathcal{O}(n) & \xrightarrow{t} & \mathcal{O}(n+1) & \to & \mathcal{O}(n+1)/t\mathcal{O}(n) & \to & 0
\end{array}$$

This diagram restricted to $D^+(f)$

$\rightsquigarrow$ The left & middle vertical maps are iso's

$\longrightarrow$ The ring map is an iso.

i.e. $H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n)) \cong H^0(X, \mathcal{O}(1)/t\mathcal{O}) = k(x) = C_{x_y}$

From this the surjectivity is reduced to show for $n=0$

$\rightsquigarrow$ $n=0$ the surjectivity of Fontaine theta map.

$$B^{\varphi=p} \xleftarrow[\cong]{\log([-])} 1+\mathfrak{m}_F \xrightarrow{\varepsilon \mapsto \log([\varepsilon^\#])} k(x) = C_{x_y}$$

Remains to show injectivity : i.e. show the kernel

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of $B^{\varphi=p^{n+1}} \twoheadrightarrow B^{\varphi=p^{n+1}} / tB^{\varphi=p^n} \longrightarrow H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n))$

is $tB^{\varphi=p^n}$.

For this, pick $s \in B^{\varphi=p^{n+1}}$ s.t. $v(s)=0$ in $H^0(X, \mathcal{O}(n+1)/t\mathcal{O}(n))$

$\boxed{\text{Goal 2}}$ $s = s_0 s_1 \cdots s_n$ with each $s_i \in B^{\varphi=p}$.

$\mathcal{O}(n+1)/t\mathcal{O}(n)$ is supported on $x$

$s$ vanishes at $x$ $\rightarrow$ some $s_i$ vanishes at $x$.

$\boxed{\text{Better than Goal}}$

$$\text{div}: \bigcup_{n\geq 0} \left(B^{\varphi=p^n} \setminus 0\right)/\mathbb{Q}_p^\times \xrightarrow[\cong]{\text{div}} \underbrace{\text{Div}^+(|Y|/\varphi^{\mathbb{Z}})}$$

$$:= \text{Div}^+(|Y|)^{\varphi}$$

$$:= \text{Div}^+(|Y_{[a,a^p)}|)$$

$\Rightarrow s_i$ is a $\mathbb{Q}_p^\times$-multiple of $t \in B^{\varphi=p}$.

$\Rightarrow s = s_0 s_1 \cdots s_n \in tB^{\varphi=p^n}$. $\boxed{\text{ES}}$

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## §3.1 Corollaries

Cor 2 (Geo explanation of FES of p-adic Hodge).

Cor 1 $\mathcal{O}(1)$ is a non-torsion element in $\text{Pic}(X)$.

Pf. Otherwise $\exists N$ s.t. $\mathcal{O}(N) \cong \mathcal{O}$

Pf. Otherwise $\exists N$ s.t. $\mathcal{O}(N) \cong \mathcal{O}$

$$\Rightarrow \underset{B^{\varphi=p^N}}{\underset{\parallel}{H^0(X, \mathcal{O}(N))}} = \underset{B^{\varphi=id}}{\underset{\parallel}{H^0(X, \mathcal{O})}}$$

$$\underset{\boxed{\text{Goal } 3}}{\underset{\uparrow}{\neq}} \qquad ⚡ \qquad \square$$

Cor 3 $H^1(X, \mathcal{O}(-1)) \neq 0$

Warning This is different from $\mathbb{P}^1_{\mathbb{Q}_p}$ : $H^1(\mathbb{P}^1_{\mathbb{Q}_p}, \mathcal{O}(-1)) = 0$.

Pf. $0 \to \mathcal{O}(-1) \xrightarrow{t} \mathcal{O} \to k(\underbrace{\infty_t}) \to 0$

the unique vanishing locus of $t$.

$$\xrightarrow{\text{LES}} \quad \underset{\underset{\mathbb{Q}_p}{\parallel}}{B^{\varphi=id}} = H^0(X, \mathcal{O}) \to H^0(X, k(\infty_t)) = C_{\infty_t} \to H^1(X, \mathcal{O}(-1)) \to \underset{\text{Coh Thm}}{\underset{\uparrow}{0}}$$

$$\Rightarrow H^1(X, \mathcal{O}(-1)) = C_{\infty_t}/\mathbb{Q}_p \neq 0 \qquad \square$$

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## §4. The Picard

Thm $\mathrm{Pic}(X) = \mathbb{Z}$.

Rmk This is the same as $\mathbb{P}^1_{\mathbb{Q}_p}$.

Regular curve $\rightsquigarrow$ the basics of divisors, line bundles still work well.

<u>What to do</u>?

$$\begin{tikzcd}
\text{Weil divisors} \arrow[r] & \mathrm{Div}(X) \arrow[r, "\text{“deg map for } \mathbb{P}^1_{\mathbb{Q}}\text{”}"] \arrow[d] & \mathbb{Z} \\
& \mathrm{Div}(X)/\mathrm{Div}_{\text{principal}}(X) \arrow[ur, dashed, "\exists!"'] & \\
& \mathrm{Pic}(X) \arrow[u, phantom, "\cong" rotate=90] &
\end{tikzcd}$$

$\longrightarrow$ denote it by "deg".

<u>Pf</u>. (1) $\deg\left(\mathrm{Div}_{\text{principal}}(X)\right) = 0$

$\left.\begin{array}{l}\text{Goal} \\ \text{reg curve thm}\end{array}\right\} \Rightarrow$ suffices to show

$f = \frac{a}{b}$ for some $a, b \in B^{\varphi = p}$.

$\Rightarrow \mathrm{div}(f) = \infty_a - \infty_b$ has deg 0.

(2) $0 \to \mathbb{Z}[\mathcal{O}(\infty_t)] \to \mathrm{Pic}(X) \to \mathrm{Pic}(\underbrace{\mathrm{Spec}\, B_e}_{D^+(t)}) \to 0$

<u>Right exactness</u> ✓

<u>Injectivity</u> Cor 1 of Coh Thm. $\left[ B_e = B\left[\frac{1}{\underbrace{\log([\varepsilon])}_{=:t}}\right]^{\varphi = id} \text{ with } E = \mathbb{Q}_p,\ \varepsilon \in 1 + m_F \setminus \{1\} \right]$

$\Rightarrow \mathrm{Pic}(X) \cong \mathbb{Z}$. ∎

## §5. Étale fundamental gp.

Q How to give finite covers? why we study it?

↳ pushforward of l.b. give v.b. which are easier to study.

most naively → Base change

$$X_E := X \times_{\mathbb{Q}_p} E$$

First properties (1) $E/\mathbb{Q}_p$ fin (sep) $\Rightarrow X_E \to X$ f. ét.

$\Rightarrow X_E$ Dedekind.

[Actually locally on $D^+(f)$, it is ~~still~~ PID]

(2) $\mathbb{Q}_p \simeq H^0(X, \mathcal{O})$

$$\Rightarrow E = H^0(X, \mathcal{O}) \otimes_{\mathbb{Q}_p} E = H^0(X_E, \mathcal{O}_{X_E})$$

$\Rightarrow X_E$ is conn.

$$\boxed{\begin{array}{l} X := X_E := X_{E,F,\pi} \\ X_{\mathbb{Q}_p,\overline{F},p} \times_{\mathbb{Q}_p} E \overset{!}{=} X_{E,\overline{F},p} \end{array}}$$

Fact $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p,F}) = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Rmk This is the same as $\mathbb{P}^1_{\mathbb{Q}_p}$.

Rmk when $F$ is only perfect & not necessarily alg closed

$$\pi_1^{ét}(X_{E,F}) = \mathrm{Gal}(\bar{E}/E) \times \mathrm{Gal}(\bar{F}/F)$$

Pf [Kedlaya 18, Simp. Conn of FF-curves]
[Weinstein 2017, $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a fund. gp]